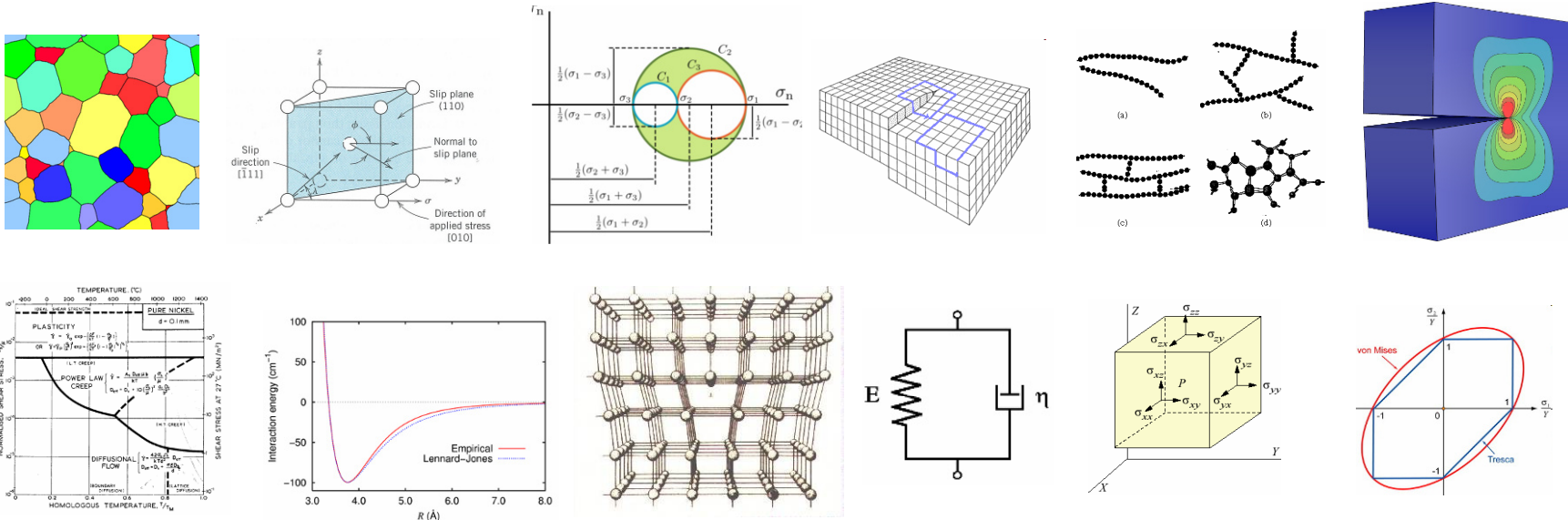# What we should know about mechanics of materials

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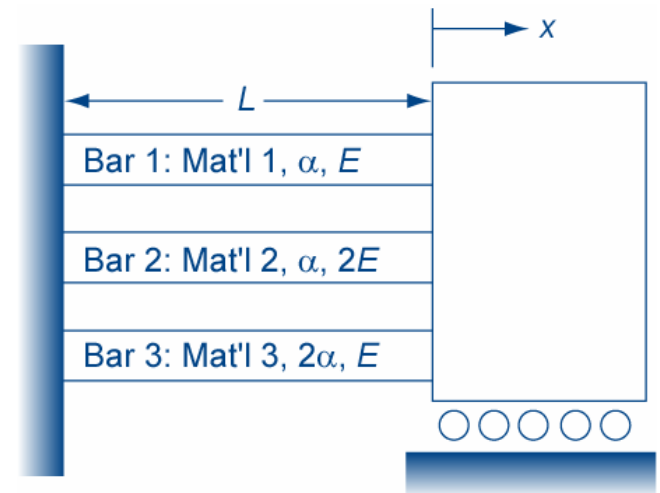Massachusetts Institute of Technology

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0. Every spontaneous process is driven by free energy minimization.

This generalizes our mechanical intuition: that forces are balanced when a system is in equilibrium.

Energy minimization easily explains mechanisms such as vacancy formation, splitting into partial dislocations, brittle fracture, solute strengthening, rubber elasticity, and many others.
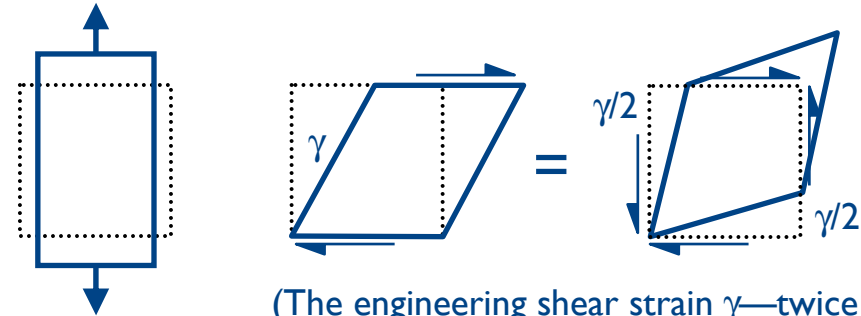
$$dG = -S\,dT + F\,dL \qquad \text{(1-D)}$$
$$dG = -S\,dT + \sigma V\,d\varepsilon \qquad \text{(3-D)}$$

Bar 1: Mat'l 1, α, E

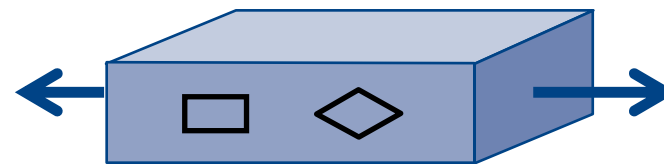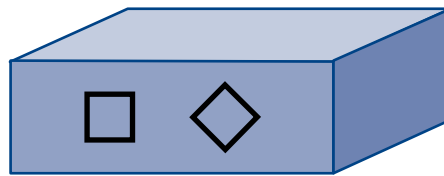Bar 2: Mat'l 2, α, $2E$

Bar 3: Mat'l 3, 2α, E

Knowing ΔT, find x:
a 50-minute forces problem,
a 10-minute strain energy problem.

1. Infinitesimal elements: *normal* strain changes side *length*, while *shear* strain changes side *angle*.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

(The engineering shear strain γ—twice the tensorial strain ε—is the angular reduction in radians.)

But in finite bodies, *dilatational* (or hydrostatic) strain changes *size*, while *deviatoric* strain changes *shape*.

$$\sigma_{\mathrm{dil}} = \frac{\sigma_{kk}}{3} = -p \qquad \boldsymbol{\sigma}_{\mathrm{dev}} = \boldsymbol{\sigma} - \boldsymbol{I}\sigma_{\mathrm{dil}}$$

2. *E* in simple Hooke's Law is restricted to uniaxial loading of long objects where lateral stresses are negligible.

$$\sigma = E\varepsilon$$

Generalized Hooke's Law works for any loading condition and geometry *of an isotropic material.*

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

Dilatational stress/strain is coupled by *K*, while deviatoric stress/strain is coupled (in the pure shear case) by *G*, both easily derived from generalized Hooke's Law. Both must be positive, constraining ν to $-1 \leq \nu \leq 0.5$.

3. Stiffness may be dominated by *enthalpy H* (metals, ceramics, polymers below T_g) or by *entropy S* (crosslinked polymers above T_g).

$$E = \left(\frac{\partial^2 U}{\partial \varepsilon^2}\right) - T\left(\frac{\partial^2 S}{\partial \varepsilon^2}\right)$$

Enthalpic stiffness correlates with density and melting temperature, decreases slightly with increasing temperature.
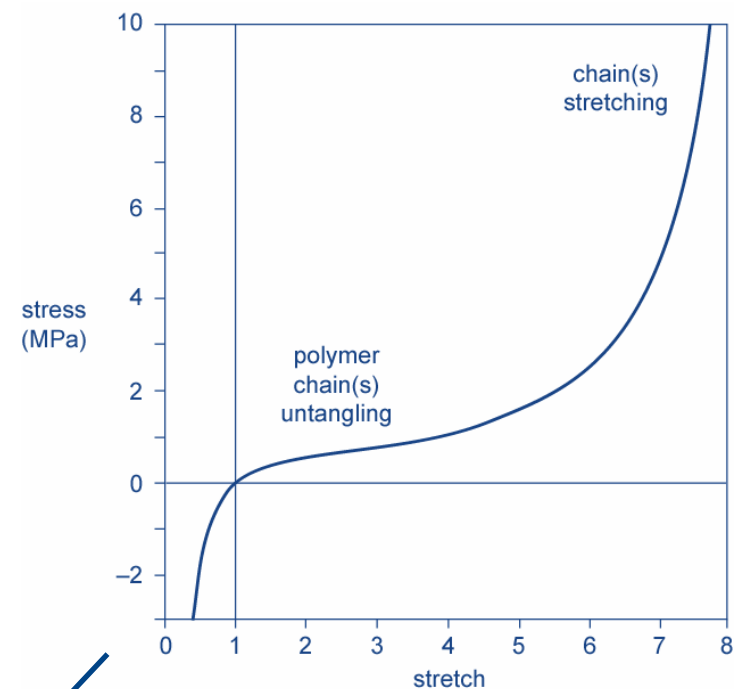
Entropic stiffness in single molecules increases linearly with temperature. (Overall stiffness increases strongly as molecule straightens and covalent, enthalpic bond stretching dominates.)

4

4. The 2nd-rank strain tensor couples undeformed and deformed vectors; the 2nd-rank stress tensor couples force and surface normal vectors.

   These tensors are symmetric because we discard rotation and translation.

   A 4th-rank tensor couples stress and strain with a maximum of 21 independent components; this tensor is symmetric because (1) applying stress and recording strain is equivalent to (2) applying strain and recording stress.

   $$\sigma = C\varepsilon$$

Caution: the compressed 6 x 6 compliance and stiffness matrices are not tensors.

5. Tensor transformations are essential for determining planes of maximum normal and shear stress; matrix eigenvalues correspond to principal stresses and strains.
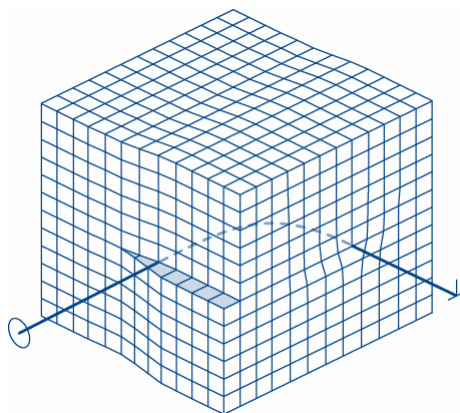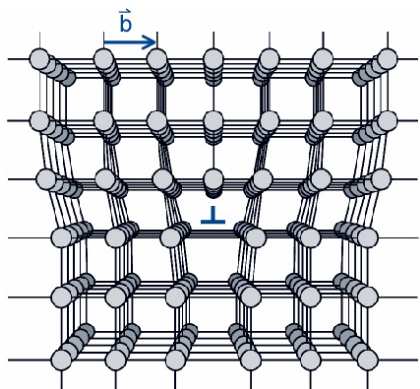
Can there be a more thrilling tensor than the direction cosine matrix a?   (***SARCASM***)

$$\sigma' = a\sigma a^{\top}$$

Mohr's circle provides a (relatively) more enjoyable graphical interpretation.
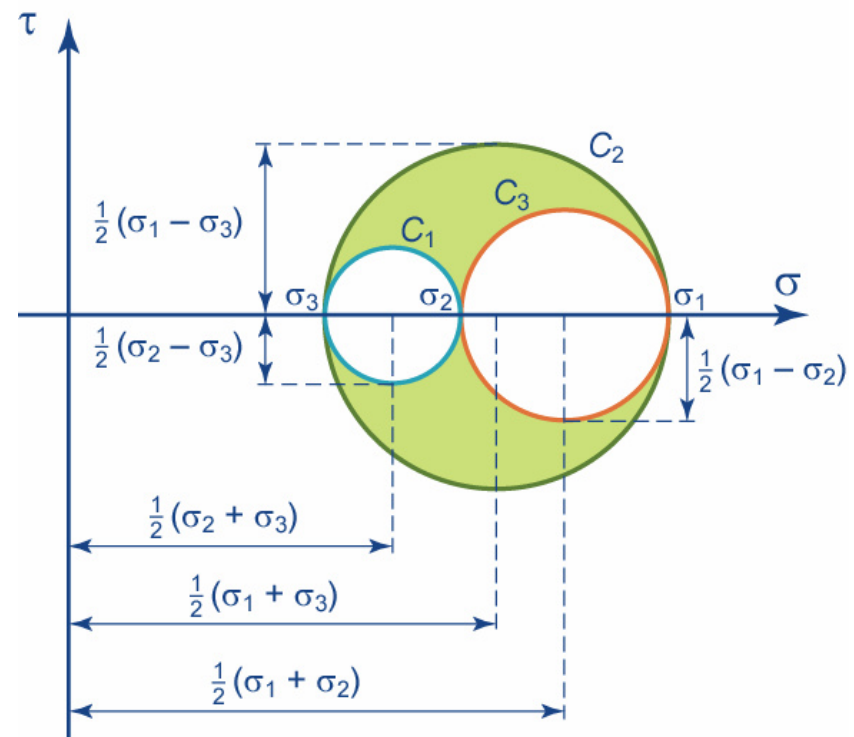
6. Dislocations are 1-D defects that carry plasticity through a crystal (with energy $\propto Gb^2$); they demarcate slipped and unslipped regions.
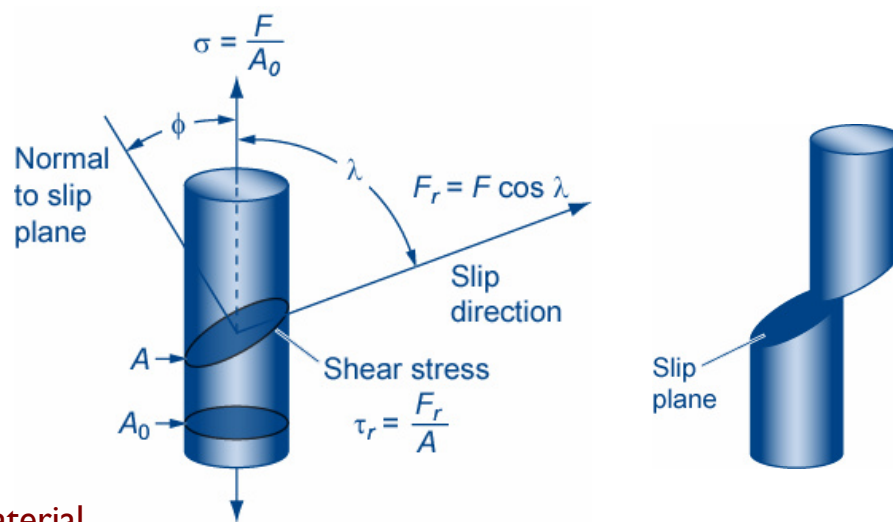

DEFECT COUNTOFF!!!

0-D: vacancies, interstitials
1-D: dislocations
2-D: grain boundaries, surfaces
3-D: precipitates

Dislocations glide at a sufficient shear stress on the most closely packed plane, in the most closely packed direction.

$$\tau_{res} = \frac{P}{A} \cos\phi \cos\lambda$$

$\sigma = \dfrac{F}{A_0}$

Normal to slip plane

$F_r = F \cos\lambda$

Slip direction

A

A_0

Shear stress

$\tau_r = \dfrac{F_r}{A}$

Slip plane

7   Caution: (111), [110] is not a slip system for any material.

7. The minimal influence of hydrostatic (compressive dilatational) stress on plastic deformation means that a zone of safety exists around $\sigma_x = \sigma_y = \sigma_z < 0$ for solid materials.
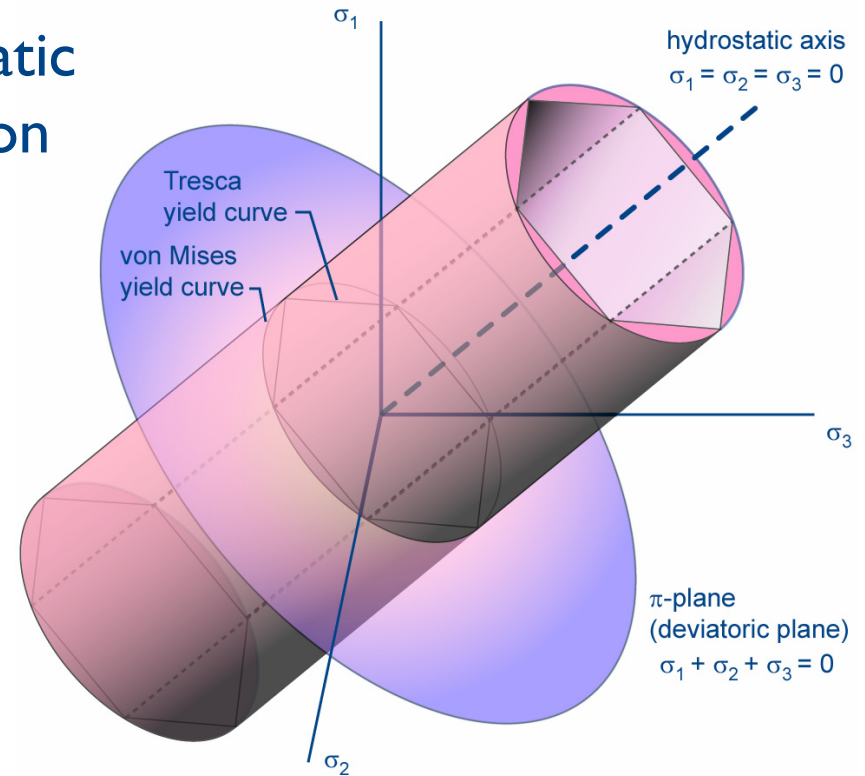
Our models of the zone boundary correspond to failure criteria.
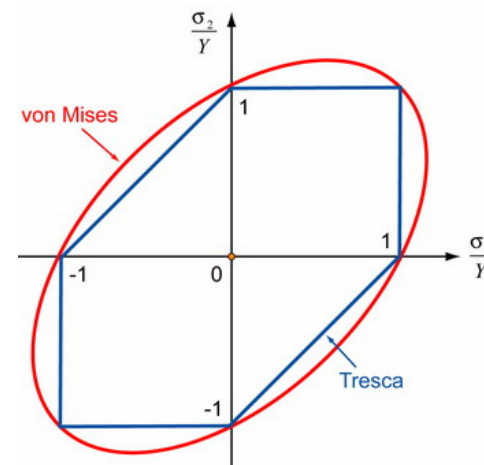
8. To strengthen crystals, inhibit dislocation motion.

   Add misfitting solute atoms that will preferentially sit in dislocation stress fields, requiring diffusion to achieve glide.

   Add dislocations by cold working that will entangle each other, generate jogs, produce sessile segments, increase length.

   Reduce grain size to suppress dislocation pile-up at grain boundaries.

   Nucleate precipitates or add particles to force dislocations to bow around or cut through.

   (Each technique has its limits.)

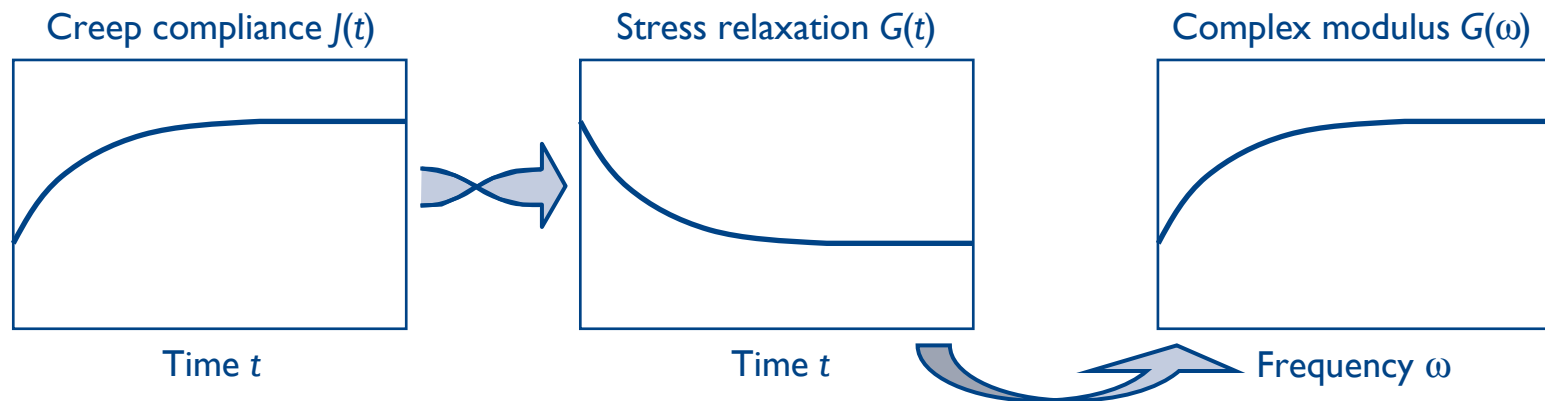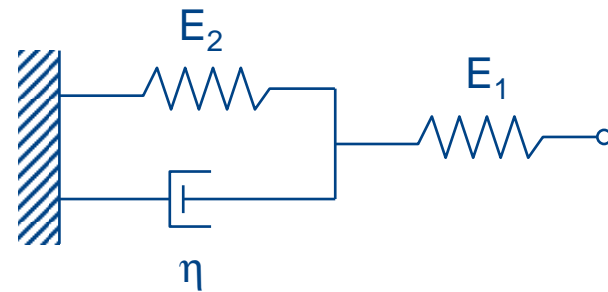## 9. There is no such thing as elasticity, only negligible plasticity.
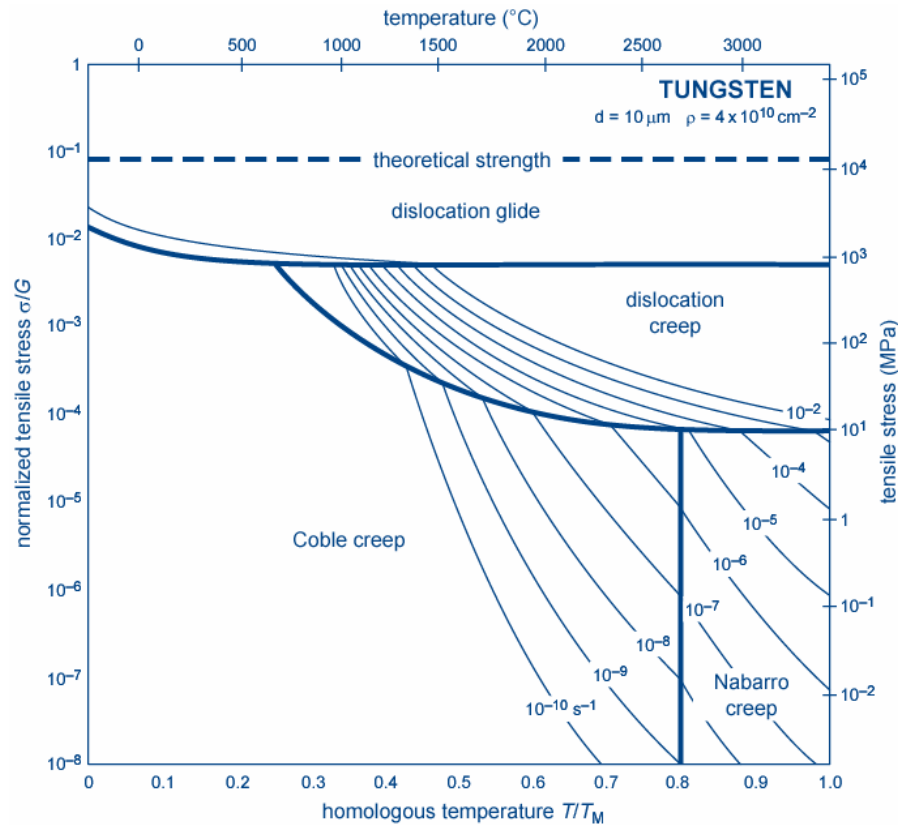
A variety of thermally activated mechanisms get in our face as the absolute temperature exceeds ~T_m/2.

$$\dot{\varepsilon} \propto \exp\left(-\frac{\Delta E}{RT}\right)\frac{\sigma^a}{d^b}$$

10   Caution: creep mechanisms may act in series or parallel.

10. Time-dependent behavior is often modeled by arrangements of springs and dashpots, enabling a complex mechanical impedance.

Springs are purely elastic (stiffness E); dashpots are purely viscous (stiffness $\eta j\omega$)

E_2

E_1

η

Creep compliance $J(t)$

Stress relaxation $G(t)$

Complex modulus $G(\omega)$

Time t

Time t

Frequency ω

The practitioner of viscoelasticity who learns Laplace transforms is a happy practitioner. Stiffness of the assembly above:

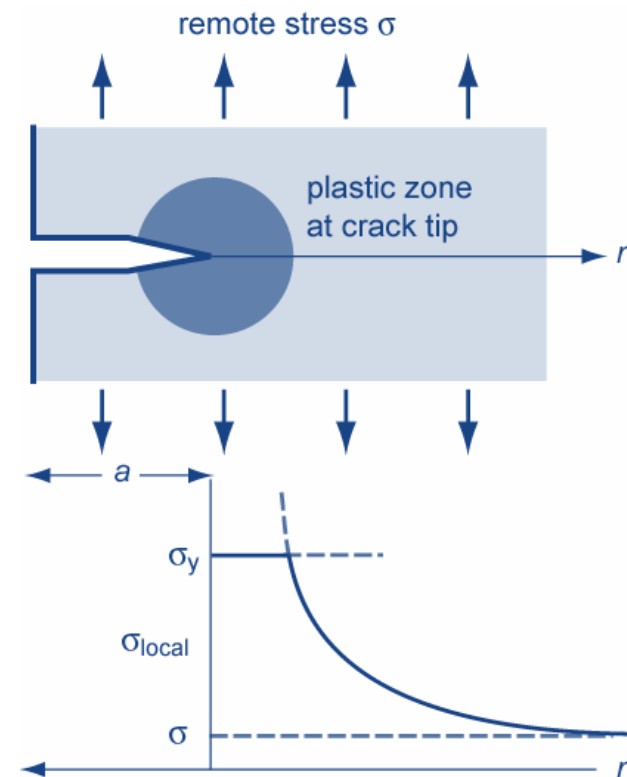$$\frac{1}{\frac{1}{E_1} + \frac{1}{E_2 + \eta j\omega}}$$

11. Brittle fracture occurs when the stress concentration at a crack tip cannot be sufficiently blunted by plastic deformation.

When the benefit of relieving strain energy more than offsets the penalty of making more new surface, look out.

$$\sigma_{\text{crit}} = \sqrt{\frac{2E\gamma}{\pi a}}$$

Fracture toughness K_C is a material property.

$$K_{\text{IC}} \propto \sigma \sqrt{\pi a_C}$$

remote stress σ

plastic zone at crack tip

r

a

σ_y

σ_{local}

σ

r

Caution: this mechanism has nothing to do with the crack reducing the load-bearing, cross-sectional area.
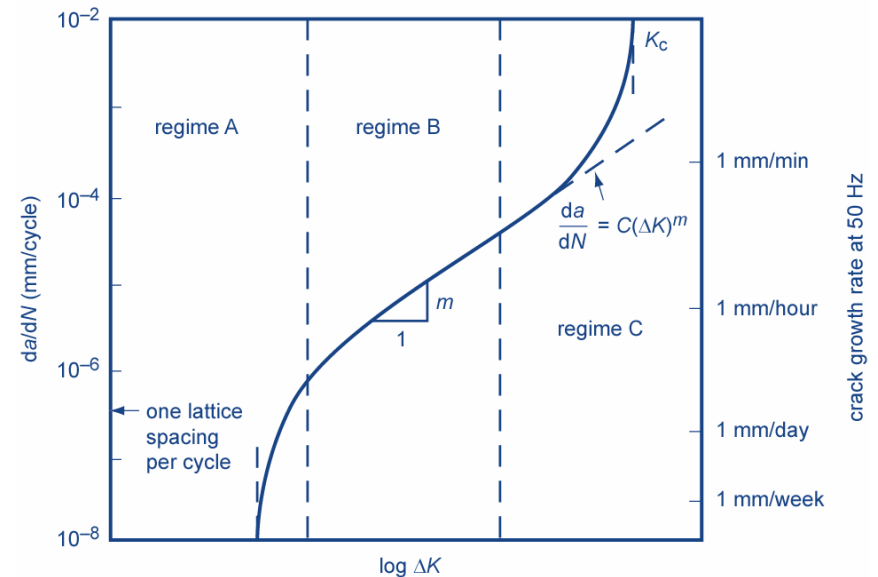
12. Repeated loading can resharpen and reblunt crack tips, enabling incremental crack propagation.

In the Paris law regime, crack growth rate depends predictably on the stress intensity factor ΔK.

$$\Delta K \propto \Delta\sigma\sqrt{\pi a}$$

$$\frac{da}{dN} \propto \Delta K^{m}$$

13. A grab bag of topics fills out our knowledge.

Composites

Pressure vessels

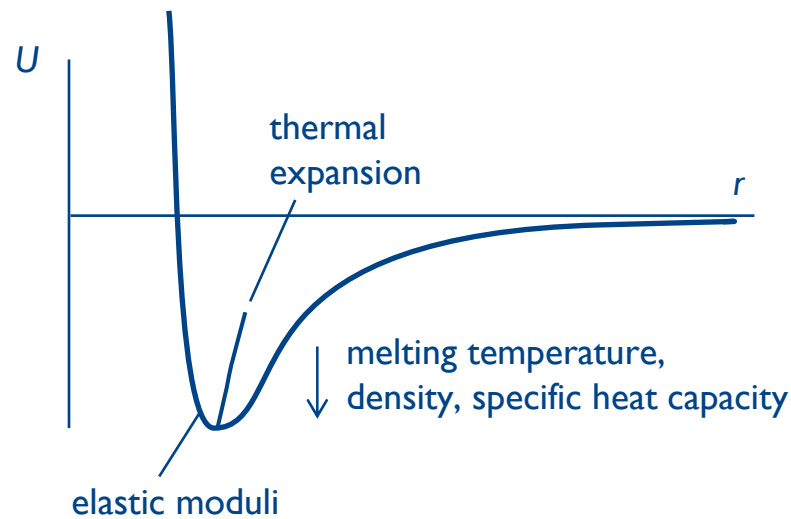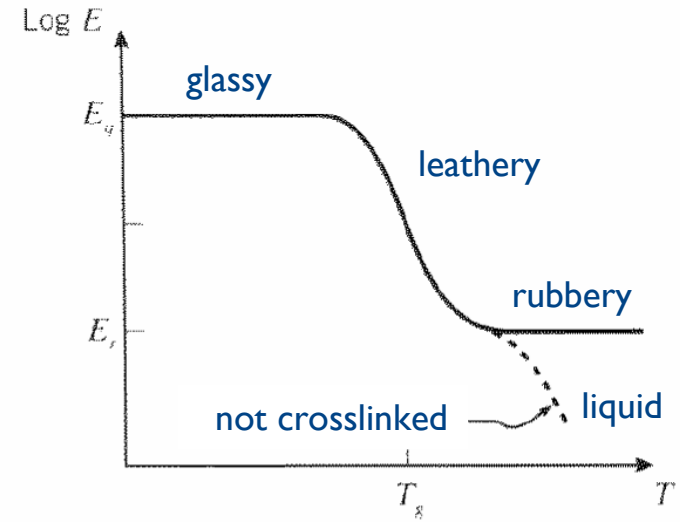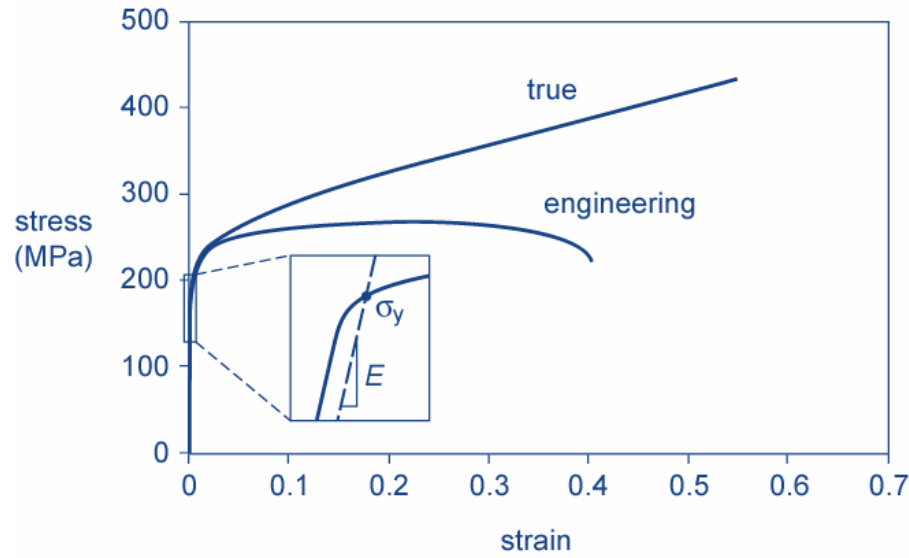Cellular solids

Elastic instability (buckling)

Further plasticity mechanisms such as twinning and GB sliding

Deformation and fracture in polymers

14. Order-of-magnitude values.

	Metals	Ceramics	Polymers
E (GPa)	100	100s	0.001–1
ν	0.33	0.25	up to 0.5
K_{IC} (MPa m$^{1/2}$)	10–100	1	1
γ (J/m^2)	1	1	1
ρ (g/cc)	3–20	3	1

15. Sketches.

Sincere thanks to:

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